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Date: December 13, 1976

Project Title: Theoretical Analysis of a Boundary Value Formulation in Linear Elasticity Using a ~~Duper~~ Dupin Coordinate System

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Project Director: Dr. Robert W. Shreeves

Sponsor: Naval Underwater Systems Center, New London, CT 06320

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Naval Underwater Systems Center
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New London, CT 06320

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Assigned to: Engineering Science and Mechanics (School/Laboratory)

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CA—3 (3/76)

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Project Title: Theoretical Analysis of a Boundary Value Formulation in Linear Elasticity Using a Duper Coordinate System

Project No: E-23-625

Project Director: Dr. Robert W. Shreeves

Sponsor: Naval Underwater Systems Center; New London, CT 06320

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- Final Report of Inventions
- Govt. Property Inventory & Related Certificate
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CA-4 (3/76)

Theoretical Analysis of a Boundary Value Formulation in Linear Elasticity Using a Dupin Coordinate System

by

Robert W. Shreeves
Georgia Institute of Technology
Atlanta, Georgia 30332

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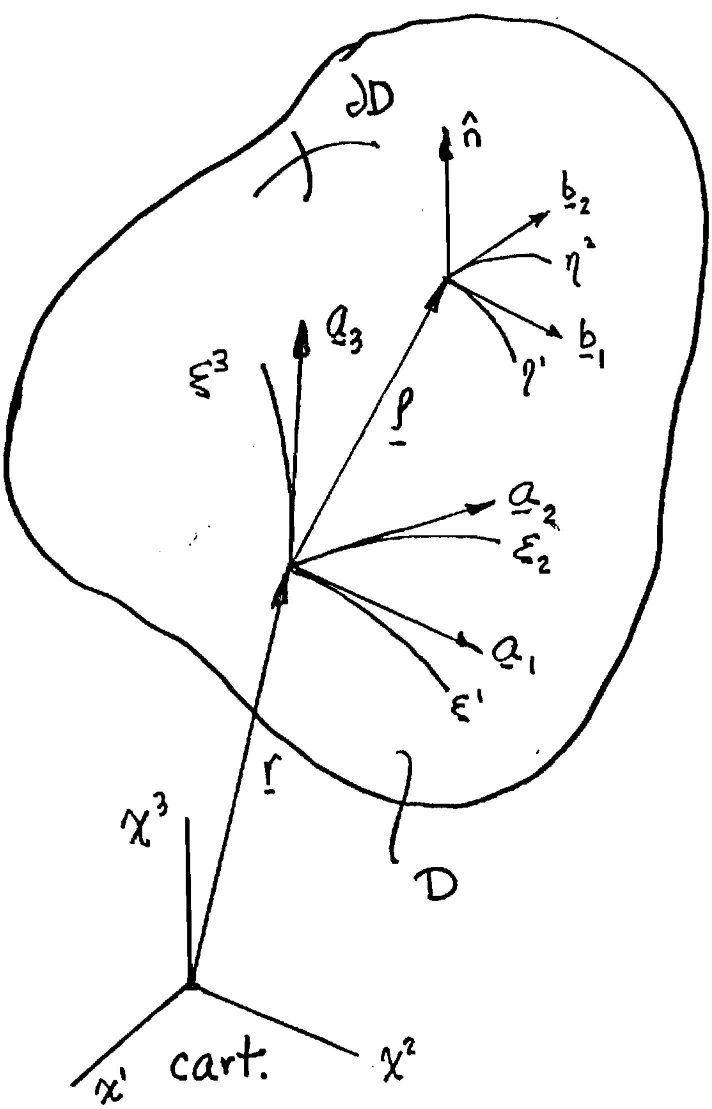
I. Introduction and Preliminaries

In the following discussion it is assumed that the reader is familiar with the use of covariant and contravariant components of vectors and tensors of at least the second order.



Let P be a point inside a body D and Q be a point on the bounding surface ∂D. Let $\xi^i = \xi^i(x^1,x^2,x^3)$ (i = 1,2, or 3) be orthogonal curvilinear coordinates for the space occupied by D. Now, in formulating a boundary value problem, BVP, one must satisfy a set of field equations, expressed either in terms of stress or displacement, throughout D subject to prescribed stress-type and or displacement-type information over the entire surface ∂D. We shall spend some effort presently to furnish a kinematically formulated field equation in terms of the orthogonal curvilinear coordinate, ξ^i.

In furnishing boundary data, it is natural to evaluate functions of ξ^i, obtained from the field equations, on ∂D where the ξ^i take on their respective boundary values. On the other hand, some advocate the choosing of surface coordinates η^1, η^2 (on ∂D) along lines of principal curvature ($\eta^3 = n$) so that at a point Q on ∂D one finds the triply-orthogonal system $\eta^1, \eta^2, \eta^3 = n$. Hence, η^α are surface coordinates for all of ∂D of D; α=1,or 2. Incidently, choosing η^α in this manner does not require the ξ^i to be orthogonal; but, we will furnish field information for such a condition because this case is more difficult to "write down" than the case in which ξ^i are general curvilinear.

The surface of the body is "connected" to the body by

$$\xi^i = \xi^i(\eta^1, \eta^2) \tag{1}$$

or

$$\underline{\rho} = \underline{\rho}(\eta^1, \eta^2)$$

In the figure, $\dot{P}$ is some convenient origin for the ξ-system and hence; functions like $f(\xi^1, \xi^2, \xi^3)$ can be written as $f[\xi^1(\eta^1,\eta^2), \xi^2(\eta^1,\eta^2), \xi^3(\eta^1,\eta^2)]$ or $\bar{f}(\eta^1,\eta^2)$ on ∂D. This maneuver may be helpful in that boundary stresses and displacements can be expressed explicitly in terms of two variables.

We are now ready to discuss the appropriate base vectors, and their derivatives, required to express the intrinsically vector elasticity equations. However, a few ground rules are necessary to get us started.

1. A dash under an index means do not sum on that index.

   Ex. $a_{\underline{i}\underline{i}}$ means a_{11} or a_{22} or a_{33}

   Ex. $a_{ijj\underline{j}} = a_{i111} + a_{i222} + a_{i333}$. Notice the third j "goes for a ride".

2. Contravarient and covariant indicies will be used for clarity but in their final form all expressions will have lowered indicies (and all components

will be physical).

3. Covariant differentiation will be indicated by a 'per'sign

Ex. $a_{i/j}$; $b_{j/k}$.

4. All unit vectors will be indentified by a carot '$\wedge$' over the letter.

Ex. $\hat{n}$ = unit normal; $\hat{i}$, $\hat{j}$, $\hat{k}$ cartesian base vectors; $\underset{\sim}{v}/|v| = \hat{v}$

5. δ^{ij}, δ_{ij}, δ^{i}_{j} will all mean the same as δ^{i}_{j}, the Kronecker delta.

6. Most symbolism will either be understood from context or will be that which the author feels the reader knows.

From the figure,

$$\underline{r} = \underline{r}(\xi^1, \xi^2, \xi^3) \tag{2}$$

$$d\underline{r} = \underline{r}_{,i}\, d\xi^i = \underline{a}_i\, d\xi^i \tag{3}$$

$$d\underline{r} \cdot d\underline{r} = (ds)^2 = \underline{a}_i \cdot \underline{a}_j\, d\xi^i d\xi^j$$

$$= g_{ij}\, d\xi^i d\xi^j \tag{4}$$

But $g_{ij} = 0$ if $i \neq j$ (ξ^i are othogonal).

$$\therefore\ (ds)^2 = g_{11}(d\xi^1)^2 + g_{22}(d\xi^2)^2 + (g_{33})(d\xi^3)^2$$

where $\quad g_{\underline{ii}} = |\, \underline{a}_i \cdot \underline{a}_i\, |$

Let
$$\left.\begin{aligned} &\underline{a}_1 \cdot \underline{a}_1 = h_1^2\,,\ \underline{a}_2 \cdot \underline{a}_2 = h_2^2\,,\ \underline{a}_3 \cdot \underline{a}_3 = h_3^2 \\ &\underline{a}_i \cdot \underline{a}_i = h_i^2 \end{aligned}\right\} \tag{5}$$

Then

$$(ds)^2 = (h_1 d\xi^1)^2 + (h_2 d\xi^2)^2 + (h_3 d\xi^3)^2 \quad . \quad . \quad . \quad (6)$$

Note: $\underline{a}_i = h_i \hat{\underline{\xi}}_i$ ($\hat{\underline{\xi}}_i$ = unit vector in ξ^i-direction) . The reciprocal vectors $\underline{a}^i$ are obtained from the following:

$$\underline{a}^1 = \frac{\underline{a}_2 \times \underline{a}_3}{\underline{a}_1 \cdot \underline{a}_2 \times \underline{a}_3} = \frac{h_2 \hat{\xi}_2 \times h_3 \hat{\xi}_3}{h_1 h_2 h_3} = \frac{h_2 h_3}{h} \hat{\xi}_1 = \frac{1}{h_1} \hat{\xi}_1 \quad . \quad . \quad (7)$$

but $\underline{a}^1 = h^1 \hat{\xi}^1 = \frac{1}{h_1} \hat{\xi}_1$ and due to the orthogonality $\hat{\xi}^1$ and $\hat{\underline{\xi}}_1$ are unit vectors in the same direction.

$$\therefore h^1 = \frac{1}{h_1}$$

and

$$\underline{a}^i = \frac{1}{h_{\underline{i}}} \hat{\xi}_i = \frac{1}{h_{\underline{i}}^2} \underline{a}_i \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In Eq. (8) we have broken the rules on free indicies but we understand the meaning. It says that the $\underline{a}_i$ and the a^i are 'nearly' self-reciprocal (the $\hat{\xi}_i$ and the $\hat{\xi}^i$ are self reciprocal) and with care we shall be permitted to lower indicies. Further,

$$\left.\begin{aligned} g_{ij} &= \underline{a}_i \cdot \underline{a}_j = h_{\underline{i}} h_{\underline{j}} \delta_{ij} \\ g^{ij} &= \underline{a}^i \cdot \underline{a}^j = \frac{1}{h_{\underline{i}}} \frac{1}{h_{\underline{j}}} \delta^{ij} \end{aligned}\right\} \quad . \quad . \quad . \quad (9)$$

For the surface ∂D we have an analogous set of relationships.

$$\underline{\rho} = \underline{\rho}(\eta^1, \eta^2) \quad . \quad . \quad . \quad . \quad (10)$$

$$d\underline{\rho} = \underline{\rho}_{,\alpha} d\eta^\alpha = \underline{b}_\alpha d\eta^\alpha \quad . \quad . \quad . \quad (11)$$

$$d\underline{\rho}\cdot d\underline{\rho} = (ds)^2 = \underline{b}_\alpha \cdot \underline{b}_\beta \, d\eta^\alpha d\eta^\beta$$

$$= E_{\alpha\beta} \, d\eta^\alpha d\eta^\beta \quad . \quad . \quad . \qquad (12)$$

where $E_{\alpha\beta} = 0 \quad (\alpha \neq \beta)$

$$\therefore (ds)^2 = E_{11} (d\eta^1)^2 + E_{22} (d\eta^2)^2$$

Let $E_{11} = \underline{b}_1 \cdot \underline{b}_1 \,, \; E_{22} = \underline{b}_2 \cdot \underline{b}_2 \,; \; E_{11} = H_1^2 , \; E_{22} = H_2^2$

$$(ds)^2 = H_1^2 (d\eta^1)^2 + H_2^2 (d\eta^2)^2 \quad . \quad . \quad . \qquad (13)$$

Note: $\underline{b}_\alpha = H_\alpha \hat{\eta}_\alpha$ ($\hat{\eta}_\alpha$ = unit vector in η^α-direction) $\underline{b}_3 = \hat{n}$, the unit normal to ∂D at Q. The reciprocal vectors $\underline{b}^\alpha$:

$$\underline{b}^1 = \frac{\underline{b}_2 \times \underline{b}_3}{\underline{b}_1 \cdot \underline{b}_2 \times \underline{b}_3} = \frac{H_2 \hat{\eta}_2 \times \hat{n}}{H_2 H_1 \cdot 1} = \frac{H_2 \hat{\eta}_1}{H_1}$$

$$\therefore \underline{b}^1 = \frac{1}{H_1} \hat{\eta}_1 \quad . \quad . \quad . \qquad (14)$$

so $$\underline{b}^\alpha = \frac{1}{H_{\underline{\alpha}}} \hat{\eta}_\alpha = \frac{1}{H_{\underline{\alpha}}^2} \underline{b}_\alpha \quad . \quad . \quad . \qquad (15)$$

Also,

$$E_{\alpha\beta} = H_{\underline{\alpha}} H_{\underline{\beta}} \delta_{\alpha\beta}$$

$$E^{\alpha\beta} = \frac{1}{H_{\underline{\alpha}}} \frac{1}{H_{\underline{\beta}}} \delta^{\alpha\beta}$$

In order to use the appropriate operators (∇, ∇^2, etc.) with the field and boundary equations we shall need derivatives of base vectors $\underline{a}_i$, ε_i, $\underline{b}_\alpha$, $\hat{\eta}_\alpha$, and $\hat{n}$.

Recall

$$\underline{a}_{i,j} = \left\{ {m \atop i\,j} \right\} \underline{a}_m = g^{mk} [ij,k]\, \underline{a}_m \quad \ldots \tag{16}$$

where

$$[ij,k] = \frac{1}{2}\left(g_{jk,i} + g_{ki,j} - g_{ij,k} \right) \quad \ldots \tag{17}$$

$$[ij,k] = \frac{1}{2}\Big[(h_{\underline{k}} h_{\underline{j}})_{,i}\, \delta_{kj} + (h_{\underline{i}} h_{\underline{k}})_{,j}\, \delta_{ik} - (h_{\underline{i}} h_{\underline{j}})_{,k}\, \delta_{ij} \Big] \quad \ldots \tag{18}$$

$$\left.\begin{aligned}
&[\underline{i}\,\underline{i},\underline{i}] = \frac{1}{2}(h_{\underline{i}}^2)_{,i} = h_{\underline{i}}\, h_{\underline{i},\underline{i}} \\
&[\underline{i}\,\underline{i},k] = -\frac{1}{2}(h_{\underline{i}}^2)_{,k} = -h_{\underline{i}}\, h_{\underline{i},k} \qquad i \neq k \\
&[\underline{i}\,j,\underline{i}] = [j\,\underline{i},\underline{i}] = \frac{1}{2}(h_{\underline{i}}^2)_{,j} = h_{\underline{i}}\, h_{\underline{i},j} \qquad i \neq j \\
&[i\,j,k] = 0 \qquad i \neq j \neq k
\end{aligned}\right\} \tag{19}$$

With (19) and (16) we get:

$$\left.\begin{aligned}
\underline{a}_{1,1} &= \frac{h_{1,1}}{h_1}\,\underline{a}_1 - \frac{h_1 h_{1,2}}{h_2^2}\,\underline{a}_2 - \frac{h_1 h_{1,3}}{h_3^2}\,\underline{a}_3 \\
\underline{a}_{2,2} &= -\frac{h_2 h_{2,1}}{h_1^2}\,\underline{a}_1 + \frac{h_{2,2}}{h_2}\,\underline{a}_2 - \frac{h_2 h_{2,3}}{h_3^2}\,\underline{a}_3 \\
\underline{a}_{3,3} &= -\frac{h_3 h_{3,1}}{h_1^2}\,\underline{a}_1 - \frac{h_3 h_{3,2}}{h_2^2}\,\underline{a}_2 + \frac{h_{3,3}}{h_3}\,\underline{a}_3
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\underline{a}_{1,2} - \underline{a}_{2,1} &= \frac{h_{1,2}}{h_1}\underline{a}_1 + \frac{h_{2,1}}{h_2}\underline{a}_2 \\
\underline{a}_{2,3} = \underline{a}_{3,2} &= \frac{h_{2,3}}{h_2}\underline{a}_2 + \frac{h_{3,2}}{h_3}\underline{a}_3 \\
\underline{a}_{3,1} = \underline{a}_{1,3} &= \frac{h_{3,1}}{h_3}\underline{a}_3 + \frac{h_{1,3}}{h_1}\underline{a}_1
\end{aligned}\right\} \qquad (20)$$

Now $\underline{a}_{i,j} = (h_{\underline{i}}\hat{\xi}_i)_{,j}$

$$\therefore h_{\underline{i}}\hat{\xi}_{i,j} = [ij,k]\frac{1}{h_{\underline{k}}}\hat{\xi}_k - h_{\underline{i},j}\hat{\xi}_i \quad . \quad . \quad . \qquad (21)$$

Therefore,

$$\left.\begin{aligned}
\hat{\xi}_{1,1} &= -\frac{h_{1,2}}{h_2}\hat{\xi}_2 - \frac{h_{1,3}}{h_3}\hat{\xi}_3 \\
\hat{\xi}_{2,2} &= -\frac{h_{2,3}}{h_3}\hat{\xi}_3 - \frac{h_{2,1}}{h_1}\hat{\xi}_1 \\
\hat{\xi}_{3,3} &= -\frac{h_{3,1}}{h_1}\hat{\xi}_1 - \frac{h_{3,2}}{h_2}\hat{\xi}_2 \\
\hat{\xi}_{1,2} &= \frac{h_{2,1}}{h_1}\hat{\xi}_2 \ , \quad \hat{\xi}_{2,1} = \frac{h_{1,2}}{h_2}\hat{\xi}_1 \\
\hat{\xi}_{2,3} &= \frac{h_{3,2}}{h_2}\hat{\xi}_3 \ , \quad \hat{\xi}_{3,2} = \frac{h_{2,3}}{h_3}\hat{\xi}_2 \\
\hat{\xi}_{3,1} &= \frac{h_{1,3}}{h_3}\hat{\xi}_1 \ , \quad \hat{\xi}_{1,3} = \frac{h_{3,1}}{h_1}\hat{\xi}_3
\end{aligned}\right\} \qquad (22)$$

We can use (16) and (19) to obtain $\underset{\sim}{b}_{\alpha,\beta}$ and $\hat{\eta}_{\alpha,\beta}$; $H_3 = 1$ and 'comma 3' means differentiation with respect to n.

$$\left.\begin{aligned}
\underset{\sim}{b}_{1,1} &= \frac{H_{1,1}}{H_1}\underset{\sim}{b}_1 - \frac{H_1}{H_2^2}H_{1,2}\underset{\sim}{b}_2 - H_1 H_{1,3}\hat{n} \\
\underset{\sim}{b}_{2,2} &= -\frac{H_2}{H_1^2}H_{2,1}\underset{\sim}{b}_1 + \frac{H_{2,2}}{H_2}\underset{\sim}{b}_2 - H_2 H_{2,3}\hat{n} \\
\underset{\sim}{b}_{3,3} &= 0 \\
\underset{\sim}{b}_{1,2} = \underset{\sim}{b}_{2,1} &= \frac{H_{1,2}}{H_1}\underset{\sim}{b}_1 + \frac{H_{2,1}}{H_2}\underset{\sim}{b}_2 \\
\underset{\sim}{b}_{2,3} = \underset{\sim}{b}_{3,2} &= \frac{H_{2,3}}{H_2}\underset{\sim}{b}_2 = -\frac{1}{r_2}\underset{\sim}{b}_2 \\
\underset{\sim}{b}_{3,1} = \underset{\sim}{b}_{1,3} &= \frac{H_{1,3}}{H_1}\underset{\sim}{b}_1 = -\frac{1}{r_1}\underset{\sim}{b}_1
\end{aligned}\right\} \tag{23}$$

where $\frac{1}{r_1}$ and $\frac{1}{r_2}$ are principal curvatures at Q.

$$\left.\begin{aligned}
H_1\hat{\eta}_{1,1} &= -\frac{H_1}{H_2}H_{1,2}\hat{\eta}_2 - H_1 H_{1,3}\hat{n} \\
H_2\hat{\eta}_{2,2} &= -\frac{H_2 H_{2,1}}{H_1}\hat{\eta}_1 - H_2 H_{2,3}\hat{n} \\
H_3\hat{\eta}_{3,3} &= 1 \cdot \hat{\eta}_{3,3} = 1 \cdot \hat{n}_{,3} = 0
\end{aligned}\right\} \tag{24}$$

$$\left.\begin{array}{l} H_1 \hat{\eta}_{1,2} = H_{2,1} \hat{\eta}_2 \; ; \; H_2 \hat{\eta}_{2,1} = H_{1,2} \hat{\eta}_1 \\ H_1 \hat{\eta}_{1,3} = H_2 \hat{\eta}_{2,3} = 0 \\ H_3 \hat{\eta}_{3,1} = \hat{n}_{,1} = H_{1,3} \hat{\eta}_1 \; ; \; H_3 \hat{\eta}_{3,2} = \hat{n}_{,2} = H_{2,3} \hat{\eta}_2 \end{array}\right\}$$

and

$$H_{1,3} = -\frac{1}{r_1} H_1 \quad ; \quad H_{2,3} = -\frac{1}{r_2} H_2$$

II. Field Equations

The kinematical formulation of the field equation of equilibrium is

$$(\lambda+\mu)\nabla\nabla\cdot\underline{u} + \mu\nabla^2\underline{u} + \rho\underline{f} = 0 \quad . \quad . \quad . \tag{25}$$

λ, μ are Lame' constants and $\underline{u}$ is the displacement. In general index form (25) is

$$(\lambda+\mu)\vartheta_{,i} + \mu g^{jk} u_{i/jk} + \rho f_i = 0 \tag{26}$$

where $\vartheta = u^i{}_{/i} = g^{im} u_{m/i}$, the 'dilatation' and u_i is the covariant component of the displacement vector $\underline{u}$. The second order Eq. (25) may be replaced by two first order equations.

$$\left.\begin{array}{r} (\lambda+2\mu)\nabla\vartheta - 2\mu\nabla\times\underline{\omega} + \rho\underline{f} = 0 \\ \nabla\cdot\underline{\omega} = 0 \end{array}\right\} \tag{27}$$

$2\underline{\omega} = \nabla \times \underline{u}$, the 'rotation vector'. To write out (25) or (26) in general orthogonal curvilinear coordinates would be somewhat brutal. Some simplification is possible:

$$\nabla \cdot \underset{\sim}{u} = \frac{1}{\sqrt{g}} \left(\sqrt{g}\, u^k \right)_{,k} = \frac{1}{h} \left(h u^k \right)_{,k} \quad \ldots \tag{28}$$

g is the $\det(g_{ij})$.

Now $\quad \underset{\sim}{u} = u^i \underset{\sim}{a}_i = u_i \underset{\sim}{a}^i = u^i h_{\underline{i}} \hat{\xi}_i = u_i \frac{1}{h_{\underline{i}}} \hat{\xi}_i$

So the physical components of $\underset{\sim}{u}$ are

$$\begin{aligned} &v^i = u^i h_{\underline{i}} \qquad ; \quad v_i = (u_i)/h_{\underline{i}} \\ &v^1 = v_1 \quad , \quad v^2 = v_2 \quad , \quad v^3 = v_3 \end{aligned} \tag{29}$$

Therefore,

$$\nabla \cdot \underset{\sim}{u} = \vartheta = \frac{1}{h} \left(h \frac{v^k}{h_{\underline{k}}} \right)_{,k} \quad \ldots \tag{30}$$

also;

$$\begin{aligned} \mu g^{jk} u_{i/jk} &= \mu \frac{1}{h_{\underline{j}}} \frac{1}{h_{\underline{k}}} \delta^{jk} u_{i/jk} = \mu \frac{1}{h_{\underline{j}}^2} u_{i/jj} \\ &= \mu \frac{1}{h_{\underline{j}}^2} \left(h_{\underline{i}} v_i \right)_{/jj} \end{aligned}$$

Eq. (26) in orthogonal coordinates becomes

$$(\lambda+\mu) h_{\underline{i}} \left[\frac{1}{h} \left(\frac{h v_k}{h_{\underline{k}}} \right)_{,k} \right]_{,i} + \mu \frac{h_i}{h_{\underline{j}}^2} \left(h_{\underline{i}} v_i \right)_{/jj} + \rho f_i = 0 \tag{31}$$

where ρf_i are the physical components of the body force. Eq. (31) is no bargain.

We shall give equilibrium in terms of stress here also.

$$\nabla \cdot \underset{\sim}{S} + \rho \underline{f} = 0 \tag{32}$$

or

$$\left.\begin{aligned} \tau^{ij}{}_{/i} + \rho f^j &= 0 \\ \text{or} \quad \tau^{i}{}_{j/i} + \rho f_j &= 0 \end{aligned}\right\} \tag{33}$$

where

$$\underset{\sim}{S} = \tau^{ij} \underline{g}_i \underline{g}_j = \tau^{ij} h_{\underline{i}} h_{\underline{j}} \hat{\xi}_i \hat{\xi}_j \,,$$

and the physical stress components are

$$\left.\begin{aligned} \sigma^{ij} &= \tau^{ij} h_{\underline{i}} h_{\underline{j}} \\ \sigma_{ij} &= \tau_{ij} \frac{1}{h_{\underline{i}}} \frac{1}{h_{\underline{j}}} \end{aligned}\right\} \tag{34}$$

for othrogonal coordinates (33) becomes

$$\frac{(h\tau^{ij})_{,i}}{h} + \tau^{im} \left\{ {j \atop mi} \right\} + \rho f^j = 0 \quad . \quad . \quad . \tag{35}$$

Or, in physical components

$$\frac{h_{\underline{j}}}{h} \left(\frac{h\sigma_{ij}}{h_{\underline{i}} h_{\underline{j}}} \right)_{,i} + \frac{h_{\underline{j}}}{h_{\underline{i}} h_{\underline{m}}} \sigma_{im} \left\{ {j \atop mi} \right\} + \rho \underline{f}_j = 0 \quad . \quad . \tag{36}$$

(36) written out for $\hat{\xi}_1$ (permute to get $\hat{\xi}_2$, $\hat{\xi}_3$ components) is

$$\frac{1}{h}\left[(\sigma_{11}h_2h_3)_{,1} + (h_1\sigma_{12}h_3)_{,2} + (h_1h_2\sigma_{13})_{,3}\right]$$
$$+ \sigma_{12}\frac{1}{h_1h_2}h_{1,2} + \sigma_{13}\frac{1}{h_1h_3}h_{1,3}$$
$$-\sigma_{22}\frac{1}{h_1h_2}h_{2,1} - \sigma_{33}\frac{1}{h_1h_3}h_{3,1} + \rho f_1 = 0 \quad \ldots \qquad (37)$$

Return now to Eq. (27).

$$2\underset{\sim}{\omega} = \nabla\times\underset{\sim}{u} = u_{j/i}\,\epsilon^{ijk}\,\underset{\sim}{a}_k$$

but

$$u_{j/i} - u_{i/j} = u_{j,i} - u_{i,j}\,'$$

$$\therefore \quad 2\underset{\sim}{\omega} = u_{j,i}\,\epsilon^{ijk}\,\underset{\sim}{a}_k \quad \ldots \qquad (38)$$

$$\epsilon^{ijk}\underset{\sim}{a}_k = \underset{\sim}{a}^i\times\underset{\sim}{a}^j = \frac{1}{h_{\underline{i}}}\frac{1}{h_{\underline{j}}}\hat{\xi}_i\times\hat{\xi}_j$$

$$\therefore \quad 2\underset{\sim}{\omega} = u_{j,i}\frac{1}{h_{\underline{i}}h_{\underline{j}}}e_{ijk}\hat{\xi}_k = 2\tilde{\omega}_k\hat{\xi}_k \qquad (39)$$

$\tilde{\omega}_k$ are physical components of $\underset{\sim}{\omega}$

$$2\tilde{\omega}_k = \frac{1}{h_{\underline{i}}h_{\underline{j}}}e_{ijk}(h_{\underline{j}}\tilde{v}_j)_{,i} \quad \ldots \qquad (40)$$

also,

$$\nabla\times\underset{\sim}{\omega} = \frac{\tilde{\omega}_{j,i}}{h_{\underline{i}}h_{\underline{j}}}e_{ijk}\hat{\xi}_k \quad \ldots \qquad (41)$$

Eq. (27) in orthogonal coordinates and physical components is

$$\left.\begin{aligned}(\lambda+2\mu) h_{\underline{k}} \vartheta_{,k} - 2\mu \frac{1}{h_{\underline{i}} h_{\underline{j}}} e_{ijk} \varpi_{j,i} + \rho f_k &= 0 \\ \frac{1}{h}\left(h \frac{1}{h_{\underline{k}}} \varpi_k\right)_{,k} &= 0\end{aligned}\right\} \tag{42}$$

Our kinematical field equation will be either (31) or (42).

III. Boundary Data

Any vector on ∂D, say $\underline{u}$, may be written in terms of position on the surface.

$$\underline{u} = \underline{u}(\eta^1, \eta^2) \quad \text{on } \partial D$$

and

$$\left.\begin{aligned}\underline{u} &= u^\alpha \underline{b}_\alpha + u^3 \hat{n} \\ &= u^\alpha H_{\underline{\alpha}} \hat{\eta}_\alpha + u^3 \hat{n} \\ &= v^\alpha \hat{\eta}_\alpha + v^3 \hat{n}\end{aligned}\right\} \quad . \quad . \quad . \tag{43}$$

$$u^3 = v^3 \quad , \quad \text{since } H_3 = 1$$

$$u^\alpha = \frac{1}{H_{\underline{\alpha}}} v^\alpha \quad ; \quad u_\alpha = H_{\underline{\alpha}} v_\alpha$$

and $\quad v_1 = v^1 \quad , \quad v_2 = v^2$

The stress vector associated with the surface whose normal is $\hat{n}(\underline{b}_3)$ is

$$\underline{t}^n = \underline{t}^3$$

$$\underline{t}^3 = \tau^{3\alpha}\underline{b}_\alpha + \tau^{33}\hat{n} \quad . \quad . \quad . \qquad (44)$$

$$\underline{t}^3 = \hat{n}\cdot\underset{\sim}{S} \quad \text{on } \partial D \quad . \quad . \quad . \qquad (45)$$

where
$$\underset{\sim}{S} = \tau^{\alpha\beta}\underline{b}_\alpha\underline{b}_\beta + \tau^{\alpha 3}(\underline{b}_\alpha\hat{n} + \hat{n}\underline{b}_\alpha) + \tau^{33}\hat{n}\hat{n} \; .$$

But $\underline{t}^3$ is to be expressed kinematically.

$$\left.\begin{aligned}\underset{\sim}{S} &= \lambda\vartheta\underset{\sim}{I} + 2\mu\underset{\sim}{E}\\ &= \lambda\vartheta\underset{\sim}{I} + \mu(\nabla\underline{u} + \underline{u}\nabla)\end{aligned}\right\} \quad . \quad . \qquad (46)$$

and $\underline{t}^3 = n \cdot \underset{\sim}{S}$, so

$$\underline{t}^3 = \lambda\vartheta\hat{n} + \mu(\hat{n}\cdot\nabla\underline{u} + \hat{n}\cdot\underline{u}\nabla) \quad . \quad . \quad . \qquad (47)$$

$\nabla\underline{u} + \underline{u}\nabla = 2\underset{\sim}{E}$, strain tensor

$\nabla\underline{u} - \underline{u}\nabla = 2\underset{\sim}{\Omega}$, rotation tensor

$$\therefore \; 2\hat{n}\cdot\underline{u}\nabla = 2\hat{n}\cdot\underset{\sim}{E} - 2\hat{n}\cdot\underset{\sim}{\Omega} \qquad (\underset{\sim}{\Omega} = -\underset{\sim}{I}\times\underline{\omega})$$

$$2\hat{n}\cdot\underline{u}\nabla = \hat{n}\cdot\nabla\underline{u} + \hat{n}\cdot\underline{u}\nabla + 2\hat{n}\cdot\underset{\sim}{I}\times\underline{\omega}$$

$$\therefore\ \hat{n}\cdot\underline{u}\nabla = \hat{n}\cdot\nabla\underline{u} + 2\hat{n}\times\underline{\omega}$$

and we know that $\hat{n}\cdot\nabla\underline{u} = \frac{\partial\underline{u}}{\partial n}$

$$\therefore\ \underline{t}^3 = \lambda\vartheta\hat{n} + 2\mu\frac{\partial\underline{u}}{\partial n} + 2\mu\,\hat{n}\times\underline{\omega} \qquad (48)$$

Now $$\frac{\partial\underline{u}}{\partial n} = \frac{\partial}{\partial n}\left(u^\alpha\underline{b}_\alpha + u^3\hat{n}\right)$$

$$= u^\alpha\underline{b}_{\alpha,3} + u^3\hat{n}_{,3} = u^\alpha\underline{b}_{\alpha,3}$$

and from (23), $$\underline{b}_{\alpha,3} = -\frac{1}{r_{\underline{\alpha}}}\underline{b}_\alpha$$

$$\therefore\ \frac{\partial\underline{u}}{\partial n} = \hat{n}\cdot\nabla\underline{u} = -\frac{u^\alpha}{r_{\underline{\alpha}}}\underline{b}_\alpha = -\frac{v_\alpha}{r_{\underline{\alpha}}}\hat{\eta}_\alpha \quad . \quad . \quad . \qquad (49)$$

also,

$$\hat{n}\times\underline{\omega} = \hat{n}\times\left(\varpi_\alpha\hat{\eta}_\alpha + \varpi_3\hat{n}\right)$$

$$= \varpi_\alpha e_{\alpha\beta}\hat{\eta}_\beta$$

where $e_{\alpha\beta}$ is the two-indexed permutation symbol. Thus, in 'Dupin Coordinates'

$$\underline{t}^3 = \lambda\vartheta\hat{n} - \frac{2\mu}{r_{\underline{\alpha}}}v_\alpha\hat{\eta}_\alpha + 2\mu e_{\alpha\beta}\varpi_\alpha\hat{\eta}_\beta \quad . \quad . \quad . \qquad (50)$$

Note: $$\hat{n}\times(\nabla\times\underline{u}) = 2\hat{n}\times\underline{\omega} = \hat{n}\cdot\underline{u}\nabla - \hat{n}\cdot\nabla\underline{u}$$

Now

$$\hat{n}\cdot\nabla\underline{u} = \frac{\partial \underline{u}}{\partial n} = -\frac{U_\alpha}{r_{\underline{\alpha}}}\hat{\eta}^\alpha \quad . \quad . \quad . \quad . \quad . \qquad (49)$$

but

$$\hat{n}\cdot\underline{u}\nabla = \left(\frac{U_{3,\alpha}}{H_{\underline{\alpha}}} + \frac{U_\alpha}{r_{\underline{\alpha}}}\right)\hat{\eta}_\alpha \quad . \quad . \quad . \qquad (51)$$

(49) and (51) give

$$\hat{n}\times(\nabla\times\underline{u}) = \left(\frac{U_{3,\alpha}}{H_{\underline{\alpha}}} + \frac{2U_\alpha}{r_{\underline{\alpha}}}\right)\hat{\eta}_\alpha \quad . \quad . \quad . \qquad (52)$$

As an alternative we can use (49) and (51) on (47) to give a suprisingly simple expression.

$$\underline{t}^3 = \lambda\vartheta\hat{n} + \mu\frac{U_{3,\alpha}}{H_{\underline{\alpha}}}\hat{\eta}_\alpha \quad . \quad . \quad . \qquad (53)$$

or

$$\underline{t}^3 = \lambda\left[\frac{1}{H}\left(H\frac{U_\alpha}{H_{\underline{\alpha}}}\right)_{,\alpha} - U_3\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right]\hat{n}$$

$$+\mu\frac{U_{3,\alpha}}{H_{\underline{\alpha}}}\hat{\eta}_\alpha \quad . \quad . \quad . \qquad (54)$$

The normal and shear stresses are

$$\sigma_{33} = \lambda\left[\frac{1}{H}\left(H\frac{U_\alpha}{H_{\underline{\alpha}}}\right)_{,\alpha} - U_3\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right] \quad . \quad . \quad . \qquad (55)$$

$$\sigma_{3\alpha} = \mu\frac{U_{3,\alpha}}{H_{\underline{\alpha}}} \quad . \quad . \quad . \quad . \qquad (56)$$

IV. Four BVP.

We state here four standard B.V.P. in elasticity. Consider only kinematical formulation and symbolically represent either Eq. (25),(31) or (42) as

$$\underset{\sim}{\mathcal{Q}} \cdot \underline{u} + \rho \underline{f} = 0 .$$

I. Displacement BVP

$$\underset{\sim}{\mathcal{Q}} \cdot \underline{u} + \rho \underline{f} = 0 \qquad \text{throughout D}$$

Given $\underline{u} = U_\alpha \hat{\eta}_\alpha + U_3 \hat{n}$ on ∂D, see (43)

II. Stress BVP

$$\underset{\sim}{\mathcal{Q}} \cdot \underline{u} + \rho \underline{f} = 0 \qquad \text{throughout D}$$

Given: $\sigma_{nn} = \lambda \left[\frac{1}{H} \left(H \frac{U_\alpha}{H_{\underline{\alpha}}} \right)_{,\alpha} - U_3 \left(\frac{1}{r_1} + \frac{1}{r_2} \right) \right]$

$\sigma_{n\alpha} = \mu \dfrac{U_{3,\alpha}}{H_{\underline{\alpha}}}$ on all ∂D, see (54)

III. Mixed BVP

$$\underset{\sim}{\mathcal{Q}} \cdot \underline{u} + \rho \underline{f} = 0 \qquad \text{throughout D}$$

Given $\underline{u} = \underline{u}_o$ on ∂D_u see BVP I

$\underline{t} = \underline{t}^3$ on ∂D_σ see BVP II

where $\partial D_u + \partial D_\sigma$ = all ∂D

IV. Mixed-Mixed BVP

$$\mathcal{Q} \cdot \underline{u} + \rho \underline{f} = 0 \qquad \text{throughout D}$$

Given (a) or (b)

(a)

σ_{nn} and υ_α on ∂D

$$\left.\begin{aligned} \sigma_{nn} &= \lambda\left[\frac{1}{H}\left(H\frac{\upsilon_\alpha}{H_\alpha}\right)_{,\alpha} - \upsilon_3\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right] \\ \underline{u}\cdot\hat{\eta}_\alpha &= \upsilon_\alpha \end{aligned}\right\} \text{ on } \partial D$$

or

(b) $\sigma_{n\alpha}$ and υ_3 on ∂D

$$\left.\begin{aligned} \sigma_{n\alpha} &= \lambda\frac{\upsilon_{3,\alpha}}{H_\alpha} \\ \hat{n}\cdot\underline{u} &= \upsilon_3 \end{aligned}\right\} \text{ on } \partial D$$

These four cases cover just about any combination of stress-type and displacement-type boundary data. Little, however, has been furnished on strictly stress formulated problems, compatibility or the Beltrami-Michell equations. It was our thought that these would be of only secondary importance with your application to electromagnetics.